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Reduced Descriptions of Hydrodynamic Turbulence

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Navier–Stokes turbulence at ordinarily large Reynolds numbers can involve excitation of perhaps 10^{18} Fourier modes. A variety of proposals for reduced description have been offered, both to gain physical insight and to make computation feasible. The present brief and idiosyncratic survey discusses Galerkin approximations, Liapunov bases, Karhunen–Loeve decompositions, and statistically based decimation techniques.

KEY WORDS: Turbulence; Karhunen–Loeve; Liapunov; decimation.

1. INTRODUCTION

Atmospheric turbulence can exhibit simultaneously a range of scales from kilometers to millimeters, so that there are the order of 10^{18} excited degrees of freedom of the velocity field. Other geophysical and astrophysical turbulent flows are similarly large systems. Laboratory turbulent flows commonly can contain the order of 10^9 excited degrees of freedom. There are strong motivations to find reduced descriptions or models of such large flows: First, complete simulation of the Navier–Stokes (NS) equations for such turbulence is not feasible on existing or foreseen computers. Second, it is desired to extract the essential physical phenomena in a compact and insightful fashion.

Two approaches to reduced description of turbulence are: (a) construction of model systems of much smaller size; (b) description by a small set of statistics. There are also hybrid approaches in which (c) flow statistics are used to construct model systems of reduced size. A very simple reduced model of turbulence can be constructed by Galerkin

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approximation: removal of all wavevector modes (or other appropriate modes) above a cutoff wavenumber. Such models can be constructed so as to conserve energy and other inviscid constants of motion, but they can lose an important dynamical property of turbulence: the dissipative action of high wavenumbers on low wavenumbers. In many computer simulation models, this problem is addressed by introducing an eddy viscosity, one of the oldest concepts in turbulence theory. Eddy viscosity is a statistical representation of the effects of the eliminated modes, and models with eddy viscosity are perhaps the simplest that combine reduction of the number of modes with statistical description.

More sophisticated models have been constructed by Karhunen–Loeve expansion, in which modal decomposition based simply on boundary conditions is replaced by decomposition into eigenfunctions of the covariance matrix of the velocity field. The hope here is that more of the physics can be included in a truncation that involves relatively few degrees of freedom. One can also envisage decomposition into Liapunov eigenfunctions so that truncations can be constructed in which only the most unstable modes of motion are retained. Finally, there are models in which there is no truncation in a mode space, but, instead, the mode density is reduced at high mode numbers. Such decimated models can be constructed simply by eliminating modes and altering the interactions of the surviving modes. But less violence is done to the statistical mechanics if, instead, interaction coefficients are unaltered, and the eliminated modes are represented by statistically determined terms in the equations of motion of the surviving modes.

Systematic attempts to incorporate statistical information into reduced models bring one face to face with the profound difficulties posed by turbulence as a problem in strongly nonequilibrium statistical mechanics. From the standpoint of kinetic theory, a highly turbulent gas exhibits one-body distribution functions that are very close to absolute equilibrium values. This is no help because the turbulent hydrodynamic fluctuations imply significant nonequilibrium many-body correlations of all orders, on hydrodynamic scales. The statistical mechanics of the NS equation, which is a more valid starting point, is strongly nonequilibrium: there is important transfer of energy between modes whose mean excitation energies differ substantially. At the same time there is strongly nonlinear interaction and the excitation of very large numbers of modes.

One consequence of the interplay of strong nonlinearity and strong disequilibrium is the simultaneous presence of order and disorder. The one-point velocity distribution in many turbulent flows is nearly Gaussian, while the one-point distribution of velocity space derivatives is highly non-Gaussian. Also, recognizable but plastic spatial and temporal structures are

present: strong vortices of various kinds and, at small spatial scales, vortex tubes and sheets. It is an open question how well such a dynamics can be described by systematic statistical procedures based on low-order moments and partial probability distributions. The constrained decimation scheme described in Section 5 promises convergent approximation sequences, but both the expected rates of convergence and the practicability of the approximations are unknown.

2. MODE TRUNCATION VERSUS MODE ELIMINATION

The incompressible NS equation may be written as

$$(\partial/\partial t - \nu\nabla^2)\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nabla p = 0 \tag{2.1}$$

where ν is kinematic viscosity, $\mathbf{u}(\mathbf{x}, t)$ is the velocity field, p is pressure divided by density, and incompressibility is expressed by the solenoidal condition $\nabla\cdot\mathbf{u} = 0$. The nonlinear advective term $\mathbf{u}\cdot\nabla\mathbf{u}$ in the NS equation, together with the pressure it induces, conserves kinetic energy $K \equiv \int |\mathbf{u}|^2\, d\mathbf{x}$. Other inviscid ($\nu = 0$) constants of motion include the circulations $\oint \mathbf{u}\cdot d\boldsymbol{l}$ about all closed circuits that move with the fluid.

Let $\mathbf{u}(\mathbf{x}, t)$ be expanded in some complete orthonormal set $\boldsymbol{\phi}^i(\mathbf{x})$ of real vector eigenfunctions of a solenoidal self-adjoint operator:

$$\mathbf{u}(\mathbf{x}, t) = \sum_i y_i(t)\, \boldsymbol{\phi}^i(\mathbf{x}) \tag{2.2}$$

Then

$$K(t) = \tfrac{1}{2}\sum_i [y_i(t)]^2 \tag{2.3}$$

and the NS equation takes the form

$$dy_i/dt + \sum_j (\nu_{ij} + L_{ij})\, y_i + \sum_{jm} A_{ijm}\, y_j\, y_m = 0 \tag{2.4}$$

Here ν_{ij}, which arises from ν, L_{ij}, which comes from elimination of pressure via incompressibility, and $A_{ijm} \equiv A_{imj}$, which arises from the advection term and pressure elimination, all are constant coefficients. If the boundary conditions are cyclic on a box and the eigenfunctions are Fourier modes, ν_{ij} is diagonal and L_{ij} vanishes.

The important property of inviscid conservation of energy by individual pair and triad interactions follows immediately from overall conservation and the fact that (2.3) is a sum of squares: If only two or

three modes are excited at an instant, then the rate of change of $K(t)$ at that instant involves contributions only from those modes. This implies the detailed conservation identities

$$L_{ij} + L_{ji} = 0, \qquad A_{ijm} + A_{jmi} + A_{mij} = 0 \tag{2.5}$$

Because of (2.5), conservative Galerkin approximations can be constructed simply by retaining only some finite subset of the y_i and deleting all terms in their equations of motion that contain y factors outside the subset. If each eigenfunction identically satisfies the boundary conditions, then the Galerkin truncations will also.

Suppose that the flow lives in a cyclic box and that a Galerkin truncation is made by discarding all Fourier modes $\mathbf{k}$ above some wavenumber k_c. If the Reynolds number is high enough, the truncation will eliminate the modes where, in the unmutilated flow, most of the viscous dissipation takes place. This means that the truncation will seriously distort the dynamics of the modes retained. Similar problems beset severe mode truncations in other geometries. There is thus a strong motivation to somehow replace simple truncation by an elimination procedure that retains the essential dynamical reaction of the eliminated modes on the modes retained.

If the excitation at time $t = 0$ is confined to modes below k_c, then, in principle, the modes above k_c, which are excited at later times, can be eliminated, without any violence to the dynamics, by repeated use of (2.4) in the form of an integral equation. Each such formal elimination of a single mode raises the degree of nonlinearity in the equations for the remaining modes. Thus, the exact elimination of large numbers of modes is unthinkable. There is also an essential physical problem: the dynamical sensitivity of high-Reynolds number turbulence. It is known from experiment, computer simulation, and theory that the detailed point-to-point and time-to-time structure of the velocity field is exceedingly sensitive to small perturbation. (In contrast, low-order statistics of the field can be robustly stable to small perturbation.) This means that the exact low-order equations obtained by formal mode elimination must be similarly sensitive. It therefore would make little sense to follow exact behavior even if it were practicable.

A variety of more or less workable approximations for the dynamical effects of eliminated high-k modes have been constructed by heuristic arguments, by use of perturbation theory and renormalized perturbation theory (RPT), by renormalization group (RNG), and by constrained decimation (CDS).[1–5] The simplest such approximations add a dynamical damping to the viscous damping in (2.4) for the retained modes. RPT,

RNG, and CDS approximations also yield a random forcing term that models the effects of dynamical sensitivity. RPT and CDS approximations yield a time-lagged dynamical damping term of the form

$$\sum_j \int_0^t \eta_{ij}(t, s)\, y_j(s)\, ds \tag{2.6}$$

3. LUMLEY'S KARHUNEN–LOEVE ANALYSIS

Over a period of years, Lumley and his co-workers have exploited the expansion of $\mathbf{u}(\mathbf{x}, t)$ for statistically stationary turbulence into orthonormal eigenfunctions $\boldsymbol{\phi}^i(\mathbf{x})$ of the one-time velocity covariance[6–8]:

$$\int \mathbf{R}(\mathbf{x}, \mathbf{x}')\, \boldsymbol{\phi}^i(\mathbf{x}')\, d\mathbf{x}' = \lambda^i \boldsymbol{\phi}^i(\mathbf{x}) \tag{3.1}$$

where

$$\mathbf{R}(\mathbf{x}, \mathbf{x}') = \langle \mathbf{u}(\mathbf{x}, t)\, \mathbf{u}(\mathbf{x}', t) \rangle$$

and $\langle \cdot \rangle$ denotes ensemble average. The eigenfunctions are numbered in order of decreasing eigenvalue λ^i. The λ^i are all positive and the mean of the kinetic energy (2.3) is simply

$$\langle K \rangle = \tfrac{1}{2} \sum_i \lambda^i \tag{3.2}$$

This expansion is dynamically natural in that, for a given ensemble of velocity fields, its truncations of given order maximize the mean kinetic energy retained, in comparison with any other orthogonal expansion. Moreover, the eigenfunctions automatically have the solenoidal property of the velocity field. The validity of these truncations in Galerkin approximations depends on how compact the exact eigenvalue spectrum is. In the case of spatially homogeneous turbulence, the eigenfunctions are simply Fourier modes and the eigenvalue spectrum is the wavevector spectrum, but arranged in order of decreasing excitation instead of increasing wavevector.

It is important to point out that, even when a small set of Lumley modes contains most of the kinetic energy, the eigenfunctions need not resemble typical flow structures in physical space. This is because phase information important to differentiating structures is not expressed by $\mathbf{R}(\mathbf{x}, \mathbf{x}')$.[6] As an illustration, consider a one-dimensional process $u(x)$ that consists of a superposition of randomly signed Gaussian pulses, all of the

same width d, randomly placed on a cyclic line segment. The Lumley eigenfunctions are Fourier modes, and truncation to a wavenumber several times d^{-1} captures most of the variance. However, the eigenfunctions do not resemble the pulse structures. The covariance $R(x, x')$ equally well describes filtered white noise. A more easily addressed shortcoming is that the decomposition does not reflect time variation. The obvious formal remedy is to decompose $\mathbf{u}(\mathbf{x}, t)$ into four-dimensional eigenfunctions of the two-time covariance of $\mathbf{u}(\mathbf{x}, t)$.[6]

An interesting application has been made by Lumley and his coworkers to the wall region of turbulent pipe flow, where the eigenfunctions of low order differ significantly from Fourier modes. In this work[7,8] a straight truncation to ten modes is enhanced by adding a simple heuristic eddy damping; representing the mean velocity field by an expression in terms of averages over the turbulent field (Reynolds stresses); and including terms expressing pressure effects from the central region of the flow, which is excluded from the calculation. The eigenfunctions are not determined self-consistently from the calculation, but instead are constructed from experimental data and full computer simulations of flows.

Numerical integrations of the truncated and enhanced amplitude equations display reasonable faithfulness to the overall boundary-layer dynamics, as determined by experiment and full computer simulation. Most interestingly, they also display temporal bursts of excitation resembling, in some respects, those observed in full simulations. Several things need more clarification: How does the behavior change as more modes are added? How diagonal is the calculated covariance matrix? How sensitive are the results to the form of the eddy damping and the representation of the effects of the central flow region? How different are the results from those of a truncation and enhancement in which the eigenfunctions are determined in a simple way from boundary conditions rather than from the empirical covariance matrix?

It would be of great interest to do a Lumley-type analysis in which the eigenfunctions are determined self-consistently. To do this, one would take some truncated starting basis, either empirical or suggested by the boundary conditions, and do an iteration procedure in which the covariance matrix computed from an ensemble of solutions at one stage is used to set the truncated representation for the next stage.

4. LIAPUNOV BASES

Several attempts have recently been made to compute the Liapunov dimension of turbulent flows at modest and moderate Reynolds numbers

by full numerical simulations of flow ensembles. The Liapunov dimension is of interest in itself. Possibly also, such analysis may help to assess the plausibility of severe truncations of the modal equations and point to apt expansion bases. A brief summary of the computer results will here be followed by some general remarks on the significance of the Liapunov analysis.

Grappin and Léorat[9] have computed Liapunov exponents and Kaplan–Yorke dimension for a simulation of randomly forced two-dimensional turbulence with a resolution of 128×128 mesh points. The number of available degrees of freedom was thus $O(10^5)$. Plots of velocity and vorticity contours showed that essentially the full available resolution was utilized by the small-scale structures of the flow. The Kaplan–Yorke dimension, in contrast, was found to be about 23. The small dimensionality here is consistent with pictures of two-dimensional turbulence in which the high-wavenumber structure consists of thin boundary layers attached to and controlled by a few large eddies, the size of the system.[10] The dynamical freedom is mostly in these large eddies.

A strongly contrasting result was found by Keefe *et al.*[11] for three-dimensional turbulence. Here computer simulations of plane Poiseuille flow were made with a resolution of 33 grid points across a channel and eight grid points each in the cyclic spanwise and flowwise directions. The Reynolds number based on pressure gradient and channel half-width was 2800 and the Kaplan–Yorke dimension was placed in the range 360–400. The total number of degrees of freedom in the calculation was about 3200. The dimensionality based on simple turbulence cascade ideas is substantially less than that, because viscous effects tie the smallest possible spatial scales to larger ones. Thus, the Kaplan–Yorke dimension that was found is not an order of magnitude less than expected by the crudest dynamical modeling. Moreover, this flow is barely turbulent; the Reynolds number is between the thresholds for instability to finite and to infinitesimal disturbances.

The Liapunov exponents, from which the Kaplan–Yorke dimension is constructed, are one measure of the divergence of initially nearby solutions of the dynamical equations. If $\mathbf{u}^1(\mathbf{x}, t)$ and $\mathbf{u}^2(\mathbf{x}, t)$ are a pair of initially neighboring solutions, then an error variance $\Delta(\mathbf{x}, t)$ may be defined by[12]

$$\boldsymbol{\delta u}(\mathbf{x}, t) = \mathbf{u}^1(\mathbf{x}, t) - \mathbf{u}^2(\mathbf{x}, t), \qquad \Delta(\mathbf{x}, t) = \langle |\boldsymbol{\delta u}(\mathbf{x}, t)|^2 \rangle \tag{4.1}$$

Liapunov exponents characterize the behavior of $\Delta(\mathbf{x}, t)$ when $\boldsymbol{\delta u}(\mathbf{x}, t)$ is infinitesimal. But the behavior of $\Delta(\mathbf{x}, t)$ for finite separation of the two solutions is also of great interest. For example, the mode amplitudes below a crossover mode number may be known in detail at initial time, while

only the spectrum is known above crossover. An important predictability problem in atmospheric dynamics is the evolution of contributions to $\Delta(\mathbf{x}, t)$ below crossover from such a finite-error start.

Even the full behavior of $\Delta(\mathbf{x}, t)$ gives a highly inadequate measure of error growth. Here are two examples. First consider a scalar field $\psi(\mathbf{x}, t)$ passively advected by a normally distributed three-dimensional prescribed velocity field according to

$$(\partial/\partial t + \mathbf{u} \cdot \nabla)\, \psi(\mathbf{x}, t) = 0 \tag{4.2}$$

Fluid elements diffuse apart in such a velocity field (even if it is frozen in time). Consequently, if the scalar field initially consists of two small blobs separated by a small distance, the blobs will eventually move chaotically away from each other. Consider now the error variance associated with the pair of fields $\psi^1(\mathbf{x}, t)$, $\psi^2(\mathbf{x}, t)$ such that each field consists of a small blob, with the blobs initially close together but nonoverlapping. It is easily seen from (4.2) that the scalar error variance is independent of time. It totally fails to capture the chaotic behavior of the system.

For a second example, return to the NS equation and suppose that the initial field consists of a single localized vortex, with the vortex position initially identical in $\mathbf{u}^1(\mathbf{x}, t)$ and $\mathbf{u}^2(\mathbf{x}, t)$. Suppose also that there is distributed weak excitation that differs between the two fields, with the result that, after long trajectories for the vortices, they are nearly in the same place in the two fields, but essentially nonoverlapping. Here $\Delta(\mathbf{x}, t)$ will be as large at the later time as if the vortices had wandered very far from each other. But if the example is taken as a model for meteorological prediction of a severe storm, it makes a great difference whether the position of the vortex is almost correct or wildly wrong.

5. CONSTRAINED DECIMATION SCHEME

The constrained decimation scheme (CDS)[5] is a mode-elimination method based on a bootstrap procedure: The eliminated modes are represented by forcing terms in generalized Langevin equations for the retained modes. The statistics of the forcing terms are determined via moment constraints from the statistics of the retained modes as the latter suffer the joint action of the forcing terms and their own self-interaction. The most important moment constraint ensures energy conservation by the total nonlinear interaction of all modes, retained and eliminated. Other constraints (they are infinite in number) serve to ensure realizability of moments and to fix systematically the moments of the joint distribution of retained mode amplitudes and forcing amplitudes in terms of the dis-

tribution of retained mode amplitudes alone. The moment constraints can be formulated so that they are exact. It therefore is possible, in principle, to obtain rigorous upper and lower bounds on quantities such as turbulent energy cascade rate by seeking maxima and minima under finite subsets of the moment constraints.

The representation of eliminated modes by constrained forcing terms makes it possible to keep correct equilibrium statistical mechanics for a truncated inviscid system, even when the decimation ratio is different for different kinds of modes. In contrast, if modes are simply discarded from the equations, no adjustment of coupling coefficients for the remaining modes can preserve the correct absolute statistical equilibrium.

There are relations between CDS and both renormalized perturbation theory (RPT) and renormalization group (RNG). In a particular limit of infinitely strong decimation (most of the modes eliminated), the CDS equations under a simple moment constraint can be treated perturbatively, and they then yield the direct-interaction approximation (DIA), a self-consistent line-renormalized perturbation approximation. In another limit, where all modes above a cutoff wavenumber within an infinite similarity spectrum range are eliminated under the same simple moment constraint, CDS yields renormalized equations for the modes below cutoff. In the renormalized equations, the eliminated modes are represented by random forcing terms together with a generalized (time-lagged) dynamical damping. A sharp difference between CDS and both RPT and RNG arises when higher approximations are constructed. As more moment constraints are imposed on the forcing terms in the Langevin equations, CDS is expected to yield realizable, converging approximations to the exact statistics of the unmutilated system.[5] In contrast, it is not known how to construct realizable, converging approximation sequences in either RPT or RNG.

In order to describe CDS with the least clutter, specialize (2.4) to the case where v_{ij} is diagonal and L_{ij} vanishes (e.g., Fourier modes in a cyclic box):

$$(\partial/\partial t + \nu_i)\, y_i + \sum_{jk} A_{ijk}\, y_j\, y_k = 0 \tag{5.1}$$

Suppose, to start, that the system is large enough (for example, homogeneous turbulence in a box very large compared to any correlation scale) that each mode y_i is statistically similar to a large number of other modes. In this case, a subset χ_i of the full set of modes y_i may be chosen such that every mode y_i is statistically similar to a χ mode and every small subset of y modes is statistically similar to some small set of χ modes. (The mode index i in χ_i runs over a subset of the values it takes in y_i.) Here

statistical similarity means that low-order moments of the chosen subset of y modes are identical in value with low-order moments of the comparison subset of χ modes, or else can be accurately obtained by interpolation among moment values of subsets of χ modes. Such statistical symmetries must be consistent with the equations of motion and also require appropriate initial conditions. The χ modes will now be called the *sample* modes, and will be treated explicitly. All the y modes that are not χ modes will be called implicit modes and will be treated only statistically.

The equations for the sample modes can be written

$$(\partial/\partial t + \nu_i)\chi_i + \sum_{jk} A_{ijk}\chi_j\chi_k + q_i = 0 \tag{5.2}$$

where

$$q_i = \sum_{jk}{}' A_{ijk}\, y_j\, y_k \tag{5.3}$$

and $\sum'_{jk}$ excludes all terms where y_j and y_k both belong to the sample set. The strategy of CDS is to deal only with the sample-set equations of motion (5.2) and to treat the internal forcing q_i as a stochastic quantity whose statistics is to be found from the statistical symmetry between y and χ modes. In the case of a small system, a formally similar treatment can be carried out by dealing with collective coordinates over a collection of similar replicas of the small systems with similar probability distributions.[13]

Two kinds of constraints control the dynamics of the sample system (5.2). First there is an infinite set of symmetry constraints on the q_i. These are constructed by using the statistical symmetries or similarities among modes, and the definitions (5.3), to form expressions for moments of the joint distribution of the χ_i and q_i in terms of moments of the χ_i alone. Examples are given below. The second class of constraints consists of an infinite set of realizability inequalities, also associated with the statistical similarities. If actual ensembles of amplitudes χ_i are evolved according to (5.2), the moments of the χ_i obviously satisfy, by construction, all the realizability inequalities that express positivity of the probability distribution of the χ_i. However, the χ_i are representative of the larger set y_i, and the moments of the y_i are expressible by statistical symmetries in terms of those of the χ_i. An important fact is that realizability of the χ_i moments does not automatically assure realizability of the y_i moments. Simple counterexamples can be given.[5] The realizability inequalities on moments of the y_i thereby represent important constraints on the χ_i and, indirectly, on the q_i.

Under certain restrictions on statistical intermittency, it can be shown that converging sequences of statistical solutions to (5.1) can be construc-

ted by solving (5.2) under increasingly large sets of the symmetry and realizability constraints, involving successively higher order moments.[5] Moreover, this procedure can be carried out in such a way that successively higher members of the standard moment-equation hierarchy associated with (5.1) are satisfied.

A symmetry constraint of key importance is

$$\langle q_i(t)\,\chi_i(t')\rangle = \sum_{jk} M_{ijk} A_{ijk} \langle \chi_j(t)\,\chi_k(t)\,\chi_i(t')\rangle \tag{5.4}$$

where $\langle\cdot\rangle$ denotes ensemble average and M_{ijk} is a multiplicity factor equal to the square root of an effective decimation ratio for triad interactions in the neighborhood of ijk. At $t=t'$, (5.4) assures conservation of ensemble-averaged kinetic energy by the nonlinear interactions. In the limit of infinite decimation ratio, (5.4) has an intimate relation to DIA. A choice of $q_i(t)$ that satisfies (5.4) and obeys a particular least-square criterion yields precisely DIA. This $q_i(t)$ does not satisfy higher order symmetry constraints.

In the limit of infinite decimation, the explicit sum in (5.2) may legitimately be treated as a perturbation. Thus, a small parameter is introduced by the act of decimation where none existed before. The perturbation analysis of (5.2) under the constraints on the q_i leads to expansions that are closely related to those of RPT. DIA is a natural lowest order approximation both in CDS at infinite decimation ratio and in RPT. In higher orders (associated with imposition of higher order symmetry and realizability constraints in CDS) the expansions differ: CDS leads to convergent sequences of approximations, while RPT does not.

The standard DIA equations may be written in generalized Langevin form as[14]

$$(\partial/\partial t + \nu_i)\, y_i(t) + \int_0^t \eta_i(t, s)\, y_i(s)\, ds + b_i(t) = 0 \tag{5.5}$$

where $y_i(t)$ is one of a complete set of modes obeying (5.1), $\eta_i(t, s)$ is a dynamical damping kernel, and $b_i(t)$ is an internal random forcing. Equation (5.5) is written for the simplest case, in which the modal covariance

$$Y_{ij}(t, t') \equiv \langle y_i(t)\, y_j(t')\rangle$$

is diagonal. The quantities $\eta_i(t, s)$ and $b_i(t)$ satisfy

$$\eta_i(t, s) = -2 \sum_{jk} A_{ijk} A_{jki} G_j(t, s)\, Y_k(t, s) \tag{5.6}$$

$$\langle b_i(t)\, b_i(t')\rangle = 2 \sum_{jk} (A_{ijk})^2\, Y_j(t, t')\, Y_k(t, t') \tag{5.7}$$

Here $Y_k(t, s) \equiv Y_{kk}(t, s)$ is the response function associated with the explicitly linear equation (5.5):

$$(\partial/\partial t + \nu_i)\, G_i(t, t') + \int_0^t \eta_i(t, s)\, G_i(s, t')\, ds = 0, \qquad G_i(t', t') = 1 \tag{5.8}$$

A decimation treatment of (5.1) may be constructed that keeps *all* the modes y_i as explicit modes by means of an artifice: A large number N of systems (5.1) are considered with identical statistics, but statistically independent of each other. Then the collection of N systems is described by collective coordinates. The decimation consists in taking an explicit (sample) set that includes one collective coordinate χ_i for each mode y_i in the original single system. This device, which may be used on systems of any size, does not reduce the size of the explicit system from its original value. Its usefulness is that it introduces high levels of exact statistical symmetry, which facilitate analysis of CDS.

The form taken by the constraint (5.4) after these manipulations is, to $O(N^{-1/2})$,

$$\langle q_i(t)\, z_i(t') \rangle = \langle \xi_i(t)\, u_i(t') \rangle + \langle \zeta_i(t)\, z_i(t') \rangle \tag{5.9}$$

where

$$\chi_i(t) = z_i(t) + N^{-1/2} u_i(t) \tag{5.10}$$

$$\xi_i \equiv \sum_{jk} A_{ijk} z_j z_k, \qquad \zeta_i \equiv \sum_{jk} A_{ijk}(u_j z_k + u_k z_j) \tag{5.11}$$

z_i and u_i are $O(1)$, and the $z_i(t)$ are statistically independent for different i. The equations of motion for $z_i(t)$ and $u_i(t)$ are

$$(\partial/\partial t + \nu_i)\, z_i(t) + \int_0^t \eta_i(t, s)\, z_i(s)\, ds + b_i(t) = 0 \tag{5.12}$$

$$(\partial/\partial t + \nu_i)\, u_i(t) + \int_0^t \eta_i(t, s)\, u_i(s)\, ds + \sum_{jk} A_{ijk} z_j z_k = 0 \tag{5.13}$$

A least-squares solution for $q_i(t)$ may be found from (5.9)–(5.13). This $q_i(t)$ is the sum of the last two terms on the left side of (5.12) with $\eta_i(t, s)$ and $b_i(t)$ given by

$$\eta_i(t, s) = \int_0^t \langle \zeta_i(t)\, z_i(s) \rangle\, U_i^{-1}(s', s \,|\, 0, t)\, ds' \tag{5.14}$$

$$b_i(t) = \int_0^t \int_0^t \langle \xi_i(t)\, u_i(s) \rangle\, Z_i^{-1}(s, s' \,|\, 0, t)\, z_i(s')\, ds\, ds' \tag{5.15}$$

where the inverse functions are defined by

$$\int_0^t Z_i(t', s)\, Z_i^{-1}(s, t''|0, t)\, ds = \delta(t' - t'')$$
$$\int_0^t U_i(t', s)\, U_i^{-1}(s, t''|0, t)\, ds = \delta(t' - t'') \tag{5.16}$$

[t', t'' in the interval $(0, t)$]. Equations (5.14) and (5.15) express a straightforward formal solution of (5.9). (The variables that take least-square values in this solution are not strictly the q_i, but the quantities $q_i + N^{-1/2}\xi_i$.)

This least-squares-decimation form of DIA equations is easily seen to be equivalent to the standard form. To show this, the moments $\langle \zeta_i(t)\, z_i(s) \rangle$ and $\langle \xi_i(t)\, u_i(s) \rangle$ which appear in (5.14) and (5.15) are evaluated by solving (5.13). The identity of $\eta_i(t, s)$, as given by (5.14), with the DIA form then follows immediately from the definition of U^{-1}. The identity of $b_i(t)$, as given by (5.15), with the DIA form can be seen by forming $\langle b_i(t)\, z_i(t') \rangle$, using the definition of Z^{-1}, and comparing with the DIA expression for this average. Although the constraint (5.9) is linear in q_i, it nevertheless fixes the covariance of b_i to DIA values, via the equations of motion (5.12) and (5.13), the latter of which is nonlinear in the z_i.

The decimation form of DIA appears complicated. It has the advantage, however, that any changes in the symmetry constraints imposed are transparently reflected in the final statistical equations. By partially relaxing the constraints, final equations can be obtained that are simpler to integrate than the full DIA equations. An example is the replacement of the detailed equations (5.9) by less restrictive constraints consisting of the equal-time constraint plus several weighted integrals over difference time. The result is the replacement of the inverse functions by different, simpler ones.

For the RNG-like application of CDS to an infinite similarity range, it is first assumed that the modes in a single system are dense. A cutoff wavenumber k_c is taken; all modes below cutoff are explicit (retained) and all above cutoff are eliminated. Any moment of explicit amplitudes χ and forcing amplitudes q may be expressed, by the definition of q, as moments of the joint distribution of explicit and implicit (eliminated) amplitudes. The latter moments, in turn, may be expressed, by the similarity laws, in terms of moments of the explicit modes alone. Thus, a set of equations can be formed that includes (1) the dynamical equations and (2) a finite subset of the moment constraints, but that involves only explicit amplitudes and their statistics. If the moment constraints consist of (5.4) alone, then q_i takes the form of a random forcing, with finite correlation time, plus a time-lagged dynamical damping. It is thus like the DIA q_i in general form.

But now the covariance of the random forcing and the form of the damping function are not determined by perturbation theory, but are directly expressed in terms of triple moments of the retained modes. Because the modes are assumed to be dense, these moments may be expressed as neighborhood averages in a single realization instead of ensemble averages.

The set of equations described above may be used to investigate the consistency of *a priori* assumptions about similarity behavior. One can also see what kinds of qualitative changes in q_i are introduced by adding higher moment constraints. It should be noted that, in contrast to traditional RNG procedures, the eliminated modes are not eliminated in successive bands, but all at once. Elimination of successive (finite or infinitesimal bands) can be carried out by CDS, but there is no reason to do so.

The cutoff within an infinite similarity range discussed above is a limiting case of more general elimination of high wavenumbers. In the problem of subgrid-scale representation for computer simulations of turbulence, it is again needed to eliminate modes above a cutoff, but, in general, there is no exact similarity, CDS can be applied to this problem by using extrapolation formulas to express the moments of eliminated modes, which appear in the moment constraints, in terms of moments of explicit modes. There will then be two sources of error: one from imposing only a finite set of moment constraints and another from the particular extrapolation formula is used.

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